Similarity Solutions of the Flow of Power Law Fluids Near an Accelerating Plate

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A similarity analysis is made of the flow of power law fluids near an accelerating flat plate by the group theoretical method.[3] This is a generalization of the works of Stokes,[4] Blasius,[2] and Watson[5] for Newtonian fluid and Bird[1] and Wells[6] for non-Newtonian fluids.

A semi-infinite body of non-Newtonian fluid is bounded on one side by a flat plate, at rest initially. At time $t = 0$, the plate is set in motion parallel to itself at a velocity of $U(t)$. It is desired to find the forms of $U(t)$ for which similarity solutions exist. The simplified momentum equation can be written as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1} \frac{\partial u}{\partial y}\right) \tag{1}$$

with the following boundary conditions:

$$y = 0 : u = U(t) \qquad y = \infty : u = 0$$

where y is the coordinate perpendicular to the plate. Two cases are possible.

Case 1

A one-parameter group of transformation is chosen in the form

$$t = A^{\alpha_1}\bar{t} \qquad y = A^{\alpha_2}\bar{y} \qquad u = A^{\alpha_3}\bar{u}$$

where α_1, α_2, α_3, and A are constants. Putting these quantities into Eq. (1), equating the powers of A on both sides, and defining $m = \alpha_3/\alpha_1$ and $b = \alpha_2/\alpha_1$, we get

$$b = [1 + (n - 1)m]/(n + 1) \tag{2}$$

The next step in this method is to find the absolute invariant under this transformation group. This can be found by noticing that

$$\frac{y}{t^{[1+(n-1)m]/(n+1)}} = \frac{\bar{y}}{\bar{t}^{[1+(n-1)m]/(n+1)}} \tag{3}$$

$$u/t^m = \bar{u}/\bar{t}^m \tag{4}$$

Therefore, the absolute invariants are

$$\eta = y/t^{[1+(n-1)m]/(n+1)} \tag{5}$$

$$F_m(\eta) = u/t^m \tag{6}$$

where m is an arbitrary constant. Equations (5) and (6) are then put into Eq. (1) and the result is

$$\nu \frac{d}{d\eta}\left(\left|\frac{dF_m}{d\eta}\right|^{n-1} \frac{dF_m}{d\eta}\right) + \frac{1 + (n - 1)m}{n + 1}\, \eta \frac{dF_m}{d\eta} - m\, F_m = 0 \tag{7}$$

It is seen that similarity solutions exist if the velocity of the wall is

$$U(t) = c_1 t^m \tag{8}$$

since then the boundary conditions will be reduced to

$$\eta = 0 : F_m = C_1 \qquad \eta = \infty : F_m = 0$$

which are independent of t.

For $n = 1$ and $m = 0$, this reduces to Stokes' solution[4] and Blasius' first case.[2] For $n = 1$ and $m = 1$, this reduces to Blasius' second case.[2] For $n = 1$, but $m \neq 0$, this becomes Watson's first case.[5] For $m = 0$ and arbitrary n, this reduces to solutions of Bird[1] and Wells.[6] All are special cases of the general form.

Case 2

Another one-parameter group of transformation is chosen in the form

$$t = \bar{t} + \beta_1 b \qquad y = e^{\beta_2 b}\bar{y} \qquad u = e^{\beta_3 b}\bar{u}$$

where β_1, β_2, β_3, and b are constants. Putting these quantities into Eq. (1) and equating the powers of e^b on both sides, we get

$$\beta_3 = n\beta_3 - (n + 1)\beta_2 \tag{9}$$

For the special case of $n = 1$, Eq. (9) becomes $\beta_2 = 0$. The absolute invariant of this group may be found by noticing that

$$u/e^{pt} = \bar{u}/e^{p\bar{t}} \tag{10}$$

where $p = \beta_3/\beta_1$. Therefore, we get

$$F_p(y) = u/e^{pt} \tag{11}$$

By putting Eq. (11) into Eq. (1), we get

$$(d^2F_p/dy^2) - (p/\nu)F_p = 0 \tag{12}$$

which is Watson's second case.[5]

For $n \neq 1$, Eq. (9) becomes

$$\beta_3 = [(n + 1)/(n - 1)]\beta_2 \tag{13}$$

Again, it is seen that

$$y/e^{qt} = \bar{y}/e^{q\bar{t}} \tag{14}$$

$$u/e^{[(n+1)/(n-1)]qt} = \bar{u}/e^{[(n+1)/(n-1)]q\bar{t}} \tag{15}$$

where $q = \beta_2/\beta_1$. The absolute invariants of this group are

$$\xi = y/e^{qt} \tag{16}$$

$$G(\xi) = u/e^{[(n+1)/(n-1)]qt} \tag{17}$$

Equations (16) and (17) are then put into Eq. (1) and the result is

$$\nu \frac{d}{d\xi}\left(\left|\frac{dG_q}{d\xi}\right|^{n-1} \frac{dGq}{d\xi}\right) + q\xi \frac{dG_q}{d\xi} - \left(\frac{n+1}{n-1}\right) q\, G_q = 0 \tag{18}$$

Again, the conclusion is that similarity solutions exist if the velocity of the wall is

$$U(t) = C_2 e^{[(n+1)/(n-1)]qt} \tag{19}$$

since then the boundary conditions will be reduced to

$$\xi = 0 : G_q = c_2 \qquad \xi = \infty : G_q = 0$$

which are independent of t.

The two transformation groups have been found to be the only two forms possible although no rigorous proof has been available in the literature.[3]

References

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